$$= \{(m+x_1)(m+x_2)(m+x_3)\cdots(m+x_n)\} \cdot \{(m+x_{n+1})\,A\} \text{ by Lemma 2}$$

$$= (m+x_1)\,(m+x_2)(m+x_3)\cdots(m+x_n)\cdot(m+x_{n+1})\,A, \text{ as desired.}$$

That is, $A(0)\cdot A(x_1)\cdot A(x_2)\cdot A(x_3)\cdots A(x_n)$ equals the $m \times m$ matrix

$$(m+x_1)(m+x_2)(m+x_3)\cdots(m+x_n)\begin{vmatrix} 1 & 1 & 1 & . & . & 1 \\ 1 & 1 & 1 & . & . & 1 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 1 & 1 & 1 & . & . & 1 \end{vmatrix}.$$

Norte. There are no concerns about non-commutativity in our algebra of matrices, because A commutes with powers of itself and with any scalar matrix c.

Note also that everything above remains true if we let all scalars come from an arbitrary ring with identity (instead of the reals).

Also solved by Brian D. Beasley, Presbyterian College, Clinton, SC; David Diminnie and Michael Taylor, Texas Instruments Inc., Dallas, TX; Bruno Salgueiro Fanego, Viveiro, Spain; Ed Gray, Highland Beach, FL; Connor Greenhalgh (student, Eastern Kentucky University), Richmond, KY; G. C. Greubel, Newport News, VA; Carl Libis, Columbia Southern University, Orange Beach, AL; David E, Manes, SUNY College at Oneonta, NY; Gail Nord, Gonzaga University, Spokane, WA; Toshihiro Shimizu, Kawasaki, Japan; Morgan Wood (student, Eastern Kentucky University), Richmond, KY, and the proposer.

- **5364:** *Proposed by Angel Plaza, Universidad de Las Palmas de Gran Canaria, Spain*

Prove that $\displaystyle\sum_{k=0}^{n}\binom{2n-2k}{n-k}\binom{2k}{k}4^{-n} = 1.$

Solution 1 by Henry Ricardo, New York Math Circle, NY

The generating function of the central binomial coefficient is well known:

$$f(x) = \frac{1}{\sqrt{1-4x}} = \sum_{k=0}^{\infty}\binom{2k}{k}x^k.$$

Applying a standard theorem on the Cauchy product of two power series,

$$\left(\sum_{i=0}^{\infty}a_i x^i\right)\cdot\left(\sum_{j=0}^{\infty}b_j x^j\right) = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}a_{n-k}b_k\right)x^n,$$

to $f^2(x)$ yields

$$\sum_{k=0}^{n}\binom{2n-2k}{n-k}\binom{2k}{k} = \text{the coefficient of } x^n \text{ in } \left(\frac{1}{\sqrt{1-4x}}\right)^2$$
$$= \text{the coefficient of } x^n \text{ in } \frac{1}{1-4x} = 4^n,$$

which proves the given identity.

Comment: The identity in the problem has been known since at least the 1930s. In her article "Counting and Recounting: The Aftermath" (*The Mathematical Intelligencer*, Vol. 6, No. 2, 1984), Marta Sved provides some references and describes a number of purely combinatorial proofs of the identity, all based in some way on the count of lattice paths.

Solution 2 by Arkady Alt, San Jose ,CA

First note that

$$\binom{-1/2}{n} = \frac{-1/2\,(-1/2-1)\ldots.(-1/2-n+1)}{n!}$$
$$= (-1)^n\cdot\frac{1\cdot 3\cdot\ldots\cdot(2n-1)}{2^n n!}$$
$$= (-1)^n\cdot\frac{(2n)!}{2^{2n}\,(n!)^2}$$
$$= \frac{(-1)^n}{4^n}\binom{2n}{n} \text{ and therefore,}$$
$$\binom{2n}{n} = (-4)^n\binom{-1/2}{n}.$$

Since,

$$\binom{2k}{k}\binom{2n-2k}{n-k} = (-4)^k\binom{-1/2}{k}(-4)^{n-k}\binom{-1/2}{n-k} = (-4)^n\binom{-1/2}{k}\binom{-1/2}{n-k},$$

we have

$$\sum_{k=0}^{n}\binom{2n-2k}{n-k}\binom{2k}{k}4^{-n} = 1 \iff \sum_{k=0}^{n}\binom{-1/2}{k}\binom{-1/2}{n-k} = (-1)^n.$$

Since $\dfrac{1}{\sqrt{1+x}} = (1+x)^{-1/2} = \sum\limits_{n=0}^{\infty}\binom{-1/2}{n}x^n$ and $\sum\limits_{n=0}^{\infty}(-1)^n x^n = \dfrac{1}{1+x}$,

we obtain

$$\left(\frac{1}{\sqrt{1+x}}\right)^2 = \frac{1}{1+x} \iff \left(\sum_{n=0}^{\infty}\binom{-1/2}{n}x^n\right)^2 = \sum_{n=0}^{\infty}(-1)^n x^n$$

$$\iff \quad \sum_{n=0}^{\infty} x^n \sum_{k=0}^{n} \binom{-1/2}{k}\binom{-1/2}{n-k} = \sum_{n=0}^{\infty} n\,(-1)^n\, x^n.$$

Hence, $\sum_{k=0}^{n} \binom{-1/2}{k}\binom{-1/2}{n-k} = (-1)^n$.

Solution 3 by Nicusor Zlota, "Traian Vuia" Technical College, Focsani, Romania

We have $\dfrac{1}{\sqrt{1-x^2}} = \sum_{n\geq 0} \binom{2n}{n} 2^{-2n} x^{2n}$.

On the other hand, we have $\dfrac{1}{1-x^2} = \sum_{n\geq 0} x^{2n}$. Squaring the first power series and comparing terms give us $\sum_{k=0}^{n} \binom{2n-2k}{n-k}\binom{2k}{k} 2^{-2n} = 1$, q.e.d.

Editor's comment : Several of those who solved this problem also commented on where variations and generalizations of it can be found. E.g., **Ulrich Abel of the Technische Hochschule Mittelhessen in Friedberg, Germany** cited the paper:

Chang, G., Xu, C., "Generalization and probabilistic proof of a combinatorial identity." **American Mathematical Monthly** 118, 175-177, (2011), and also a paper of his which was published in 2015 that further generalizes notions used in the Chang and Xu paper.

Ulrich Abel, Vijay Gupta, and Mircea Ivan, "A generalization of a combinatorial identity by Change and Xu," **Bulletin of Mathematical Sciences,** published by Springer, ISSN 1664-3607. This paper can also be seen at Springer's open line access site < SpringerLink.com > .

Another citation was given by **Moti Levy, of Rehovot Israel.** He mentioned that in <u>Concrete Mathematics</u>, by Graham, Knuth, and Patashnik (second edition) the problem is solved in Section 5.3, "Tricks of the trade," pages 186-187 . **And Carl Libis of Columbia Southern University, Orange Beach, AL** cited http://math.stackexchange.com/questions/687221/proving-sum-k-0n2k-choose-k2n-2k-choose-n-k-4n/688370688370

In addition, **Bruno Salgueirio Fanego of Viveiro, Spain** stated that a probabilistic interpretation of the problem can be found in <http://mathes.pugetsound.edu/~mspivey/AltConvRepring.pdf>. He went on to say that: more generally, it can be demonstrated that, for any real l, $\sum_{k=0}^{n} \binom{2n-2k-l}{n-k}\binom{2k+l}{k} 4^{-n} = 1$ (see: http://arxiv.org/pdf/1307.6693.pdf) and that for any integer $m \geq 2$,

$$\sum_{k_1\cdot k_2\cdots k_m=n} \binom{2k_1}{k_1}\binom{2k_2}{k_2}\cdots\binom{2k_m}{k_m} 4^{-n} = \frac{\Gamma\left(\frac{m}{2}+n\right)}{n!\,\Gamma\left(\frac{m}{2}\right)},$$